GENERATING FUNCTIONS AND RECURRENCE RELATIONS

Recurrence Relations

Suppose $a_0, a_1, a_2, \ldots, a_n, \ldots,$is an infinite sequence.
A recurrence recurrence relation is a set of equations

$$a_n = f_n(a_{n-1}, a_{n-2}, \ldots, a_{n-k}). \quad (1)$$

The whole sequence is determined by (6) and the values of $a_0, a_1, \ldots, a_{k-1}$.

Linear Recurrence

Fibonacci Sequence

$$a_n = a_{n-1} + a_{n-2} \qquad n \geq 2.$$

$a_0 = a_1 = 1$.

$b_n = |B_n| = |\{x \in \{a, b, c\}^n : aa \text{ does not occur in } x\}|.$

$b_1 = 3 : \ a\ b\ c$

$b_2 = 8 : \ ab\ ac\ ba\ bb\ bc\ ca\ cb\ cc$

$b_n = 2b_{n-1} + 2b_{n-2} \qquad n \geq 2.$

$$b_n = 2b_{n-1} + 2b_{n-2} \qquad n \geq 2.$$

Let

$$B_n = B_n^{(b)} \cup B_n^{(c)} \cup B_n^{(a)}$$

where $B_n^{(\alpha)} = \{x \in B_n : \ x_1 = \alpha\}$ for $\alpha = a, b, c$.

Now $|B_n^{(b)}| = |B_n^{(c)}| = |B_{n-1}|$. The map $f : B_n^{(b)} \to B_{n-1}$,

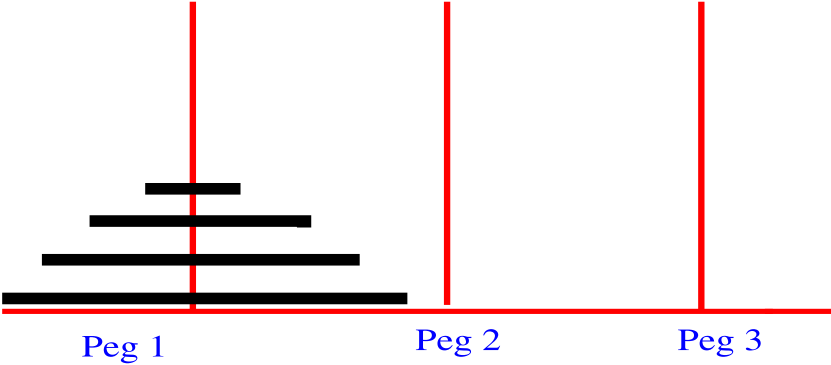
$$f(bx_2x_3 \ldots x_n) = x_2x_3 \ldots x_n \text{ is a bijection.}$$

$B_n^{(a)} = \{x \in B_n : \ x_1 = a \text{ and } x_2 = b \text{ or } c\}$. The map $g : B_n^{(a)} \to B_{n-1}^{(b)} \cup B_{n-1}^{(c)}$,

$$g(ax_2x_3 \ldots x_n) = x_2x_3 \ldots x_n \text{ is a bijection.}$$

Hence, $|B_n^{(a)}| = 2|B_{n-2}|$.

Towers of Hanoi



H_n is the minimum number of moves needed to shift n rings from Peg 1 to Peg 2. One is not allowed to place a larger ring on top of a smaller ring.

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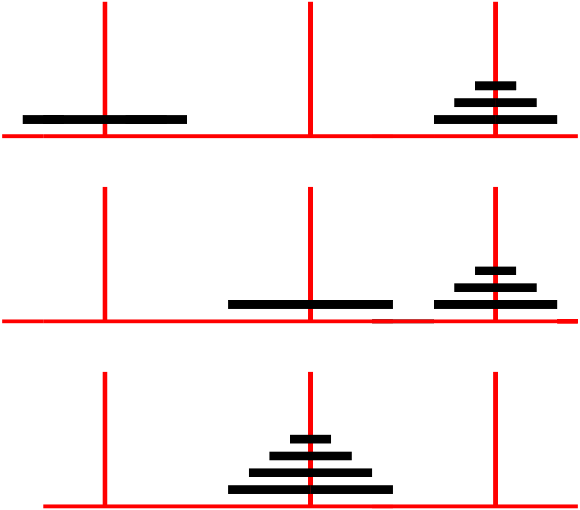


H_{n-1} moves

1 move

H_{n-1} moves

A has n dollars. Everyday A buys one of a Bun (1 dollar), an Ice-Cream (2 dollars) or a Pastry (2 dollars). How many ways are there (sequences) for A to spend his money?
Ex. BBPIIPBI represents “Day 1, buy Bun. Day 2, buy Bun etc.”.

$$\begin{aligned} u_n &= \text{number of ways} \\ &= u_{n,B} + u_{n,I} + u_{n,P} \end{aligned}$$

where $u_{n,B}$ is the number of ways where A buys a Bun on day 1 etc.

$u_{n,B} = u_{n-1},\ u_{n,I} = u_{n,P} = u_{n-2}$.
So

$$u_n = u_{n-1} + 2u_{n-2},$$

and

$$u_0 = u_1 = 1.$$

If $a_0, a_1, \ldots, a_n$ is a sequence of real numbers then its **(ordinary) generating function** $a(x)$ is given by

$$a(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n + \cdots$$

and we write

$$a_n = [x^n]a(x).$$

For more on this subject see Generatingfunctionology by the late Herbert S. Wilf. The book is available from https://www.math.upenn.edu// wilf/DownldGF.html

$a_n = 1$

$$a(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$$

$a_n = n + 1$.

$$a(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots$$

$a_n = n$.

$$a(x) = \frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots$$

Generalised binomial theorem:

$a_n = \binom{\alpha}{n}$

$$a(x) = (1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}.$$

$a_n = \binom{m+n-1}{n}$

$$a(x) = \frac{1}{(1-x)^m} = \sum_{n=0}^{\infty} \binom{-m}{n} (-x)^n = \sum_{n=0}^{\infty} \binom{m+n-1}{n} x^n.$$

General view.

Given a recurrence relation for the sequence (a_n), we

(a) Deduce from it, an equation satisfied by the generating function $a(x) = \sum_n a_n x^n$.

(b) Solve this equation to get an explicit expression for the generating function.

(c) Extract the coefficient a_n of x^n from $a(x)$, by expanding $a(x)$ as a power series.

Solution of linear recurrences

$$a_n - 6a_{n-1} + 9a_{n-2} = 0 \qquad n \geq 2.$$

$a_0 = 1, a_1 = 9$.

$$\sum_{n=2}^{\infty}(a_n - 6a_{n-1} + 9a_{n-2})x^n = 0. \tag{2}$$

$$\begin{aligned}
\sum_{n=2}^{\infty} a_n x^n &= a(x) - a_0 - a_1 x \\
&= a(x) - 1 - 9x. \\
\sum_{n=2}^{\infty} 6a_{n-1} x^n &= 6x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} \\
&= 6x(a(x) - a_0) \\
&= 6x(a(x) - 1). \\
\sum_{n=2}^{\infty} 9a_{n-2} x^n &= 9x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} \\
&= 9x^2 a(x).
\end{aligned}$$

$$a(x) - 1 - 9x - 6x(a(x) - 1) + 9x^2 a(x) = 0$$

or

$$a(x)(1 - 6x + 9x^2) - (1 + 3x) = 0.$$

$$\begin{aligned} a(x) &= \frac{1+3x}{1-6x+9x^2} = \frac{1+3x}{(1-3x)^2} \\ &= \sum_{n=0}^{\infty}(n+1)3^n x^n + 3x\sum_{n=0}^{\infty}(n+1)3^n x^n \\ &= \sum_{n=0}^{\infty}(n+1)3^n x^n + \sum_{n=0}^{\infty} n3^n x^n \\ &= \sum_{n=0}^{\infty}(2n+1)3^n x^n. \end{aligned}$$

$$a_n = (2n+1)3^n.$$

Fibonacci sequence:

$$\sum_{n=2}^{\infty}(a_n - a_{n-1} - a_{n-2})x^n = 0.$$

$$\sum_{n=2}^{\infty} a_n x^n - \sum_{n=2}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

$$(a(x) - a_0 - a_1 x) - (x(a(x) - a_0)) - x^2 a(x) = 0.$$

$$a(x) = \frac{1}{1 - x - x^2}.$$

$$
\begin{aligned}
a(x) &= -\frac{1}{(\xi_1 - x)(\xi_2 - x)} \\
&= \frac{1}{\xi_1 - \xi_2}\left(\frac{1}{\xi_1 - x} - \frac{1}{\xi_2 - x}\right) \\
&= \frac{1}{\xi_1 - \xi_2}\left(\frac{\xi_1^{-1}}{1 - x/\xi_1} - \frac{\xi_2^{-1}}{1 - x/\xi_2}\right)
\end{aligned}
$$

where

$$\xi_1 = -\frac{\sqrt{5}+1}{2} \text{ and } \xi_2 = \frac{\sqrt{5}-1}{2}$$

are the 2 roots of

$$x^2 + x - 1 = 0.$$

Therefore,

$$
\begin{aligned}
a(x) &= \frac{\xi_1^{-1}}{\xi_1 - \xi_2} \sum_{n=0}^{\infty} \xi_1^{-n} x^n - \frac{\xi_2^{-1}}{\xi_1 - \xi_2} \sum_{n=0}^{\infty} \xi_2^{-n} x^n \\
&= \sum_{n=0}^{\infty} \frac{\xi_1^{-n-1} - \xi_2^{-n-1}}{\xi_1 - \xi_2} x^n
\end{aligned}
$$

and so

$$
\begin{aligned}
a_n &= \frac{\xi_1^{-n-1} - \xi_2^{-n-1}}{\xi_1 - \xi_2} \\
&= \frac{1}{\sqrt{5}} \left(\left(\frac{\sqrt{5}+1}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right).
\end{aligned}
$$

Inhomogeneous problem

$$a_n - 3a_{n-1} = n^2 \qquad n \geq 1.$$

$a_0 = 1$.

$$\begin{aligned}
\sum_{n=1}^{\infty}(a_n - 3a_{n-1})x^n &= \sum_{n=1}^{\infty} n^2 x^n \\
\sum_{n=1}^{\infty} n^2 x^n &= \sum_{n=2}^{\infty} n(n-1)x^n + \sum_{n=1}^{\infty} nx^n \\
&= \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} \\
&= \frac{x+x^2}{(1-x)^3} \\
\sum_{n=1}^{\infty}(a_n - 3a_{n-1})x^n &= a(x) - 1 - 3xa(x) \\
&= a(x)(1-3x) - 1.
\end{aligned}$$

$$
\begin{aligned}
a(x) &= \frac{x+x^2}{(1-x)^3(1-3x)} + \frac{1}{1-3x} \\
&= \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D+1}{1-3x}
\end{aligned}
$$

where

$$
\begin{aligned}
x + x^2 \cong\ & A(1-x)^2(1-3x) + B(1-x)(1-3x) \\
& + C(1-3x) + D(1-x)^3.
\end{aligned}
$$

Then

$$
A = -1/2,\ B = 0,\ C = -1,\ D = 3/2.
$$

So

$$a(x) = \frac{-1/2}{1-x} - \frac{1}{(1-x)^3} + \frac{5/2}{1-3x}$$
$$= -\frac{1}{2}\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} \binom{n+2}{2} x^n + \frac{5}{2}\sum_{n=0}^{\infty} 3^n x^n$$

So

$$a_n = -\frac{1}{2} - \binom{n+2}{2} + \frac{5}{2}3^n$$
$$= -\frac{3}{2} - \frac{3n}{2} - \frac{n^2}{2} + \frac{5}{2}3^n.$$

General case of linear recurrence

$$a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = u_n, \qquad n \geq k.$$

$u_0, u_1, \ldots, u_{k-1}$ are given.

$$\sum (a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} - u_n)\, x^n = 0$$

It follows that for some polynomial $r(x)$,

$$a(x) = \frac{u(x) + r(x)}{q(x)}$$

where

$$q(x) = 1 + c_1 x + c_2 x^2 + \cdots + c_k x^k = \prod_{i=1}^{k} (1 - \alpha_i x)$$

and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the roots of $p(x) = 0$ where $p(x) = x^k q(1/x) = x^k + c_1 x^{k-1} + \cdots + c_0$.

Products of generating functions

$$a(x) = \sum_{n=0}^{\infty} a_n x^n,\ b(x)) = \sum_{n=0}^{\infty} b_n x^n.$$

$$\begin{aligned}
a(x)b(x) &= (a_0 + a_1 x + a_2 x^2 + \cdots) \times \\
&\qquad (b_0 + b_1 x + b_2 x^2 + \cdots) \\
&= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + \\
&\qquad (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots \\
&= \sum_{n=0}^{\infty} c_n x^n
\end{aligned}$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Derangements

$$n! = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}.$$

Explanation: $\binom{n}{k} d_{n-k}$ is the number of permutations with exactly k cycles of length 1. Choose k elements ($\binom{n}{k}$ ways) for which $\pi(i) = i$ and then choose a derangement of the remaining $n - k$ elements.

So

$$\begin{aligned} 1 &= \sum_{k=0}^{n} \frac{1}{k!} \frac{d_{n-k}}{(n-k)!} \\ \sum_{n=0}^{\infty} x^n &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} \frac{d_{n-k}}{(n-k)!} \right) x^n. \end{aligned} \tag{3}$$

Let

$$d(x) = \sum_{m=0}^{\infty} \frac{d_m}{m!} x^m.$$

From (3) we have

$$\begin{aligned} \frac{1}{1-x} &= e^x d(x) \\ d(x) &= \frac{e^{-x}}{1-x} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\frac{(-1)^k}{k!} \right) x^n. \end{aligned}$$

So

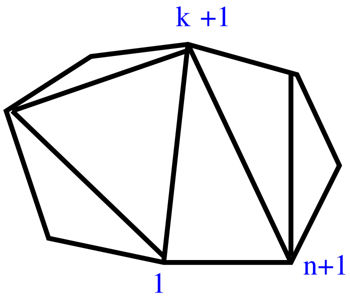
$$\frac{d_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Triangulation of n-gon

Let

$$\begin{aligned} a_n &= \text{number of triangulations of } P_{n+1} \\ &= \sum_{k=0}^{n} a_k a_{n-k} \qquad n \geq 2 \end{aligned} \tag{4}$$

$a_0 = 0$, $a_1 = a_2 = 1$.



Explanation of (4):

$a_k a_{n-k}$ counts the number of triangulations in which edge $1, n+1$ is contained in triangle $1, k+1, n+1$.

There are a_k ways of triangulating $1, 2, \ldots, k+1, 1$ and for each such there are a_{n-k} ways of triangulating $k+1, k+2, \ldots, n+1, k+1$.

$$x + \sum_{n=2}^{\infty} a_n x^n = x + \sum_{n=2}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n.$$

But,

$$x + \sum_{n=2}^{\infty} a_n x^n = a(x)$$

since $a_0 = 0, a_1 = 1$.

$$\begin{aligned} \sum_{n=2}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n \\ &= a(x)^2. \end{aligned}$$

So

$$a(x) = x + a(x)^2$$

and hence

$$a(x) = \frac{1+\sqrt{1-4x}}{2} \text{ or } \frac{1-\sqrt{1-4x}}{2}.$$

But $a(0) = 0$ and so

$$\begin{aligned} a(x) &= \frac{1-\sqrt{1-4x}}{2} \\ &= \frac{1}{2} - \frac{1}{2}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^{2n-1}} \binom{2n-2}{n-1} (-4x)^n\right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^n. \end{aligned}$$

So

$$a_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

Exponential Generating Functions

Given a sequence $a_n, n \geq 0$, its exponential generating function (e.g.f.) $a_e(x)$ is given by

$$a_e(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

$$a_n = 1, n \geq 0 \text{ implies } a_e(x) = e^x.$$

$$a_n = n!, n \geq 0 \text{ implies } a_e(x) = \frac{1}{1-x}$$

Products of Exponential Generating Functions

Let $a_e(x), b_e(x)$ be the e.g.f.'s respectively for $(a_n), (b_n)$ respectively. Then

$$c_e(x) = a_e(x)b_e(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!} \right) x^n$$
$$= \sum_{k=0}^{n} \frac{c_n}{n!} x^n$$

where

$$c_n = \binom{n}{k} a_k b_{n-k}.$$

Interpretation

Suppose that we have a collection of labelled objects and each object has a “size” k, where k is a non-negative integer. Each object is labelled by a set of size k.

Suppose that the number of labelled objects of size k is a_k.

Examples:

(a): Each object is a directed path with k vertices and its vertices are labelled by $1, 2, \ldots, k$ in some order. Thus $a_k = k!$.

(b): Each object is a directed cycle with k vertices and its vertices are labelled by $1, 2, \ldots, k$ in some order. Thus $a_k = (k-1)!$.

Now take example (a) and let $a_e(x) = \frac{1}{1-x}$ be the e.g.f. of this family. Now consider

$$c_e(x) = a_e(x)^2 = \sum_{n=0}^{\infty}(n+1)x^n \text{ with } c_n = (n+1) \times n!.$$

c_n is the number of ways of choosing an object of weight k and another object of weight $n - k$ and a partition of $[n]$ into two sets A_1, A_2 of size k and labelling the first object with A_1 and the second with A_2.

Here $(n+1) \times n!$ represents taking a permutation and choosing $0 \leq k \leq n$ and putting the first k labels onto the first path and the second $n - k$ labels onto the second path.

We will now use this machinery to count the number s_n of permutations that have an even number of cycles all of which have odd lengths:

Cycles of a permutation

Let $\pi : D \to D$ be a permutation of the finite set D. Consider the digraph $\Gamma_\pi = (D, A)$ where $A = \{(i, \pi(i)) : \; i \in D\}$. Γ_π is a collection of vertex disjoint cycles. Each $x \in D$ being on a unique cycle. Here a cycle can consist of a loop i.e. when $\pi(x) = x$.

Example: $D = [10]$.

i	1	2	3	4	5	6	7	8	9	10
$\pi(i)$	6	2	7	10	3	8	9	1	5	4

The cycles are $(1, 6, 8), (2), (3, 7, 9, 5), (4, 10)$.

In general consider the sequence $i, \pi(i), \pi^2(i), \ldots,$.

Since D is finite, there exists a first pair $k < \ell$ such that $\pi^k(i) = \pi^\ell(i)$. Now we must have $k = 0$, since otherwise putting $x = \pi^{k-1}(i) \neq y = \pi^{\ell-1}(i)$ we see that $\pi(x) = \pi(y)$, contradicting the fact that π is a permutation.

So i lies on the cycle $C = (i, \pi(i), \pi^2(i), \ldots, \pi^{k-1}(i), i)$.

If j is not a vertex of C then $\pi(j)$ is not on C and so we can repeat the argument to show that the rest of D is partitioned into cycles.

Now consider

$$a_e(x) = \sum_{m=0}^{\infty} \frac{(2m)!}{(2m+1)!} x^{2m+1}$$

Here

$$a_n = \begin{cases} 0 & n \text{ is even} \\ (n-1)! & n \text{ is odd} \end{cases}$$

Thus each object is an odd length cycle C, labelled by $[|C|]$.

Note that

$$\begin{aligned} a_e(x) &= \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right) - \left(\frac{x^2}{2} + \frac{x^4}{4} + \cdots\right) \\ &= \log\left(\frac{1}{1-x}\right) - \frac{1}{2}\log\left(\frac{1}{1-x^2}\right) \\ &= \log\sqrt{\frac{1+x}{1-x}} \end{aligned}$$

Now consider $a_e(x)^\ell$. The coefficient of x^n in this series is $\frac{c_n}{n!}$ where c_n is the number of ways of choosing an ordered sequence of ℓ cycles of lengths $a_1, a_2, \ldots, a_\ell$ where $a_1 + a_2 + \cdots + a_\ell = n$. And then a partition of $[n]$ into $A_1, A_2, \ldots, A_\ell$ where $|A_i| = a_i$ for $i = 1, 2, \ldots, \ell$. And then labelling the ith cycle with A_i for $i = 1, 2, \ldots, \ell$.

We looked carefully at the case $\ell = 2$ and this needs a simple inductive step.

It follows that the coefficient of x^n in $\frac{a_e(x)^\ell}{\ell!}$ is $\frac{c_n}{n!}$ where c_n is the number of ways of choosing a set (unordered sequence) of ℓ cycles of lengths $a_1, a_2, \ldots, a_\ell$ ...

What we therefore want is the coefficient of x^n in $1 + \frac{a_e(x)^2}{2!} + \frac{a(x)^4}{4!} + \cdots$.

Now

$$\sum_{k=0}^{\infty} \frac{a_e(x)^{2k}}{k!} = \frac{e^{a_e(x)} + e^{-a_e(x)}}{2} = \frac{1}{2}\left(\sqrt{\frac{1+x}{1-x}} + \sqrt{\frac{1-x}{1+x}}\right)$$

$$= \frac{1}{\sqrt{1-x^2}}$$

Thus

$$s_n = n![x^n]\frac{1}{\sqrt{1-x^2}} = \binom{n}{n/2}\frac{n!}{2^n}$$

Exponential Families

- P is a set referred to a set of pictures.
- A card C is a pair C, p, where $p \in P$ and S is a set of labels. The weight of C is $n = |S|$.
  If $S = [n]$ then C is a standard card.
- A hand H is a set of cards whose label sets form a partition of $[n]$ for some $n \geq 1$. The weight of H is n.
- $C' = (S', p)$ is a re-labelling of the card $C = (S, p)$ if $|S'| = |S|$.
- A deck $\mathcal{D}$ is a finite set of standard cards of common weight n, all of whose pictures are distinct.
- An exponential family $\mathcal{F}$ is a collection $\mathcal{D}_n, n \geq 1$, where the weight of $\mathcal{D}_n$ is n.

Given $\mathcal{F}$ let $h(n,k)$ denote the number of hands of weight n consisting of k cards, such that each card is a re-labelling of some card in some deck of $\mathcal{F}$.
(The same card can be used for re-labelling more than once.)
Next let the hand enumerator $\mathcal{H}(x,y)$ be defined by

$$\mathcal{H}(x,y) = \sum_{\substack{n\geq 1\\ k\geq 0}} h(n,k)\frac{x^n}{n!}y^k, \qquad (h(n,0) = \mathbf{1}_{n=0}).$$

Let $d_n = |\mathcal{D}_n|$ and $\mathcal{D}(x) = \sum_{n=1}^{\infty} \frac{d_n}{n!}x^n$.

Theorem

$$\mathcal{H}(x,y) = e^{y\mathcal{D}(x)}. \tag{5}$$

Example 1: Let $P = \{directed\ cycles\ of\ all\ lengths\}$.

A card is a directed cycle with labelled vertices.

A hand is a set of directed cycles of total length n whose vertex labels partition $[n]$ i.e. it corresponds to a permutation of $[n]$.

$d_n = (n-1)!$ and so

$$\mathcal{D}(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = \log\left(\frac{1}{1-x}\right)$$

and

$$\mathcal{H}(x,y) = \exp\left\{y \log\left(\frac{1}{1-x}\right)\right\} = \frac{1}{(1-x)^y}.$$

Let $\left[{n \atop k}\right]$ denote the number of permutations of $[n]$ with exactly k cycles. Then

$$\sum_{k=1}^{n} \left[{n \atop k}\right] y^k = \left[\frac{x^n}{n!}\right] \frac{1}{(1-x)^y}$$
$$= n! \binom{y+n-1}{n}$$
$$= y(y+1)\cdots(y+n-1).$$

The values $\left[{n \atop k}\right]$ are referred to as the Stirling numbers of the first kind.

Example 2: Let $P = \{[n], n \geq 1\}$.
A card is a non-empty set of positive integers.
A hand of k cards is a partition of $[n]$ into k non-empty subsets.
$d_n = 1$ for $n \geq 1$ and so

$$\mathcal{D}(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1$$

and

$$\mathcal{H}(x, y) = e^{y(e^x - 1)}.$$

So, if $\left\{ {n \atop k} \right\}$ is the number of partitions of $[n]$ into k parts then

$$\left\{ {n \atop k} \right\} = \left[\frac{x^n}{n!} \right] \frac{(e^x - 1)^k}{k!}.$$

The values $\left\{ {n \atop k} \right\}$ are referred to as the Stirling numbers of the second kind.

Proof of (5)**:** Let $\mathcal{F}', \mathcal{F}''$ be two exponential families whose picture sets are disjoint. We merge them to form $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ by taking all d'_n cards from the deck $\mathcal{D}'_n$ and adding them to the deck $\mathcal{D}''_n$ to make a deck of $d'_n + d''_n$ cards.

We claim that

$$\mathcal{H}(x, y) = \mathcal{H}'(x, y)\mathcal{H}''(x, y). \tag{6}$$

Indeed, a hand of $\mathcal{F}$ consists of k' cards of total weight n' together with $k'' = k - k'$ cards of total weight $n'' = n - n'$. The cards of $\mathcal{F}'$ will be labelled from an n'-subset S of $[n]$. Thus,

$$h(n, k) = \sum_{n', k'} \binom{n}{n'} h'(n', k')h''(n - n', k - k').$$

But,

$$\mathcal{H}'(x,y)\mathcal{H}''(x,y) = \left(\sum_{n',k'} h(n',k')\frac{x^{n'}}{n'!}y^{k'}\right)\left(\sum_{n'',k''} h(n'',k'')\frac{x^{n''}}{n''!}y^{k''}\right)$$
$$= \sum_{n,k}\left(\frac{n!}{n'(n-n')!}h(n',k')h(n'',k'')\right)\frac{x^n}{n!}y^k.$$

This implies (6).

Now fix positive weights r, d and consider an exponential family $\mathcal{F}_{r,d}$ that has d cards in deck $\mathcal{D}_r$ and no other non-empty decks. We claim that the hand enumerator of $\mathcal{F}_{r,d}$ is

$$\mathcal{H}_{r,d}(x,y) = \exp\left\{\frac{ydx^r}{r!}\right\}. \tag{7}$$

We prove this by induction on d.

Base Case $d = 1$: A hand consists of $k \geq 0$ copies of the unique standard card that exists. If $n = kr$ then there are

$$\binom{n!}{r!r!\cdots r!} = \frac{n!}{(r!)^k}$$

choices for the labels of the cards. Then

$$h(kr, k) = \frac{1}{k!}\frac{n!}{(r!)^k}$$

where we have divided by $k!$ because the cards in a hand are unordered. If r does not divide n then $h(n,k) = 0$.

Thus,

$$\mathcal{H}_{r,1}(x,y) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{n!}{(r!)^k} \frac{x^n}{n!} y^k$$
$$= \exp\left\{\frac{yx^r}{r!}\right\}$$

Inductive Step: $\mathcal{F}_{r,d} = \mathcal{F}_{r,1} \oplus \mathcal{F}_{r,d-1}$. So,

$$\mathcal{H}_{r,d}(x,y) = \mathcal{H}_{r,1}(x,y)\mathcal{H}_{r,d-1}(x,y)$$
$$= \exp\left\{\frac{yx^r}{r!}\right\} \exp\left\{\frac{y(d-1)x^r}{r!}\right\}$$
$$= \exp\left\{\frac{ydx^r}{r!}\right\},$$

completing the induction.

Now consider a general deck $\mathcal{D}$ as the union of disjoint decks $\mathcal{D}_r, r \geq 1$. then,

$$\mathcal{H}(x, y) = \prod_{r \geq 1} \mathcal{H}_r(x, y) = \prod_{r \geq 1} \exp\left\{\frac{ydx^r}{r!}\right\} = e^{y\mathcal{D}(x)}.$$